LIFTING π_1- INJECTIVE SURFACES IMMERSED IN 3-MANIFOLDS : A BRIEF SURVEY (Low-Dimensional Topology of Tomorrow)

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LIFTING π_1-INJECTIVE SURFACES IMMERSED IN 3-MANIFOLDS: A BRIEF SURVEY

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ABSTRACT. This is a brief, non-technical survey of the group-theoretical concept of LERFness (also called subgroup separability), the topological problem of lifting an essential immersion to an embedding, and the interplay between these two ideas through the last few decades. Some key definitions and significant results are given, as well as recent development and current conjectures.

The purpose of this survey paper is two-fold: first,to introduce a few group theoretical notions that have proved helpful in an important problem in low-dimensional topology, and second, to review briefly the historical development and progress made by both topologists and group theorists. My hope is that this survey will help the reader understand the importance of the topic at hand and point to appropriate papers and articles for further exploration of the subject.

1. INTRODUCTION

One of the basic problems in low-dimensional topology has been the following: given a π_1-injective immersion of an $n-1$ manifold S into an n-manifold M, can one lift the immersion to an embedding of S into some finite-degree covering space $\hat{M}$ of M? This "lifting problem" has proved quite useful in the study of compact surfaces since loops (1-dimensional manifolds) immersed in a surface has a lot to say about the topology of the given surface. For dimension 2, the problem has been solved affirmatively for many years now: more specifically, we have the following:

Theorem 1.1. *Suppose $f\colon S^1 \to S$ is an immersion of the circle into a compact surface S such that $f_*\colon \pi_1(S^1) \to \pi_1(S)$ is injective. Then, there is a finite-degree covering space $\hat{S}$ of S such that f lifts to an embedding $\hat{f}\colon S^1 \to \hat{S}$.*

The proof of this theorem was originally given in [24], in which Scott actually proved many other significant results, as will be mentioned

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later in this survey. In particular, the above theorem is an immediate corollary of the fact that all closed surface groups and free groups have the so-called LERF property (to be defined later), a sufficient condition to guarantee that every π_1-injective immersion lifts to an embedding in a finite cover. In Section 5, I will briefly summarize how Scott proved this very important result.

In this brief, non-technical, and historical survey of the topological question of lifting immersions, I will demonstrate how group theory and topology have interacted with each other, contributing to the development in both fields. I will begin by stating the what and the why of the topological-geometric problem and then define some key group-theoretical terms and concepts that are crucial in understanding important results. I will then cover the historical development of these ideas and conclude by stating some recent development, conjectures, and future challenges in the theory of 3-manifolds, thus making this article appropriate for a conference with a theme like "Low-Dimensional Topology of Tomorrow."

2. The Basic Topological Problem

Here is the main topological problem we are focusing on in this paper. Let us refer to this as the "Lifting Problem" for short.

Problem 2.1. Suppose $f\colon S \to M$ is a proper immersion of a compact surface into a compact 3-dimensional manifold M such that $f_*\colon \pi_1(S) \to \pi_1(M)$ is injective. Then, is there a finite-degree covering space $\hat{M}$ of M such that f lifts to an embedding $\hat{f}\colon S \to \hat{M}$?

This problem is significant for several reasons. Just as immersed and embedded loops play an important role in the theory of compact surfaces, immersed and embedded compact surfaces have been very helpful in 3-manifold theory. In particular, if an immersed surface S in M can be lifted to an embedding in a finite cover $\hat{M}$, then $\hat{M}$ is a Haken manifold, which, according to the celebrated theorem of Waldhausen, is rigid, i.e., homotopy equivalence implies homeomorphism. This would make M a virtually Haken manifold. This problem, therefore, has close connection with the following conjectures in low-dimensional topology and geometry.

Conjectures 2.2. *If M is a compact irreducible 3-manifold with an infinite fundamental group, then*
(1) M is a virtually Haken manifold ("Virtually Haken Conjecture");
(2) M is virtually $\mathbb{Z}$-representable (i.e., there exists a finite cover $\hat{M}$

of M such that $\beta_1 = rk(H_1(\hat{M})) > 0$) ("Positive Betti Number Conjecture").

Here, the second conjecture implies the first.

One way to attack the above-mentioned lifting problem has been to use the theory of groups. It is rather interesting to see the interaction between these two fields of mathematics as they have made a lot of contribution to each other. (See, for example, [26] for various techniques involving both group theory and geometry.) The topological/geometric problem was first solved using a group-theoretical concept, and later geometric techniques and topology were used to give many counterexamples in combinatorial group theory.

We now turn to the group-theoretical ideas crucial to the solution of the above-mentioned "Lifting Problem." As suggested in Section 5, historically the answers have been

- "Maybe,"
- "Probably,"
- "Yes, for many manifolds M,"
- "Definitely not for other manifolds," and
- "There seems to be a common *bad* ingredient whenever the answer is *no*."

3. Approach Using Group Theory

As mentioned in the previous section, the Lifting Problem seeks to find an embedding from a given immersion. The difference between the two is exactly what is causing the problem. It is self-intersection. In other words, if we can somehow eliminate all self-intersections of the immersed surface S in M by going up to some finite-degree covering space $\hat{M}$ of M, the problem has the positive answer. Any such (non-trivial) self-intersection produces a loop represented by some γ in $\pi_1(M)$ which is just a path (not a loop) in S. Now, since the immersion is π_1-injective, $\pi_1(S)$ can be considered a subset of $\pi_1(M)$, and such γ is an element of $\pi_1(M)\backslash\pi_1(S)$. Hence, a natural way to approach this problem is to *separate* such γ by some finite-index subgroup of $\pi_1(M)$. If we can do this to all "problem-causing" elements (i.e., generators of these self-intersections), then we will be able to produce a finite-index subgroup $\pi_1(\hat{M})$, in which all self-intersections have been eliminated, i.e., the immersion f lifts to an embedding $\hat{f}$ in $\hat{M}$, a finite-degree covering space of M. This naturally gives rise to the concept of "subgroup separability" of $\pi_1(M)$. The key idea here is captured in the following definition:

Definition 3.1. A group G is called LERF (subgroup separable) if for every finite-generated subgroup H of G and for ever $\gamma \in G \backslash H$, there is a finite-index subgroup G' of G such that $S \subset G'$ but $\gamma \notin G'$.

The following theorem provides a key link between the Lifting Problem and the LERF property. The general idea of the proof is sketched above, before the definition of LERF-ness.

Theorem 3.2. *Let M be a closed 3-manifold. If $\pi_1(M)$ is LERF, then any π_1-injective immersion of a compact surface S lifts to an embedding in a finite-degree covering space $\hat{M}$ of M.*

Before discussing the LERF property of various 3-manifold groups, we first list some basic concepts related to subgroup separability and define a few topological terms in order to understand some recent (and not so recent) results stated in Sections 5 and 6.

4. Key Definitions and Concepts

Here are some definitions preferred by algebraists but also helpful to topologists. For all these definitions, let G be a group with identity e. The first is a very general definition.

Definition 4.1. Let P be some property. G is said to be *residually P* if for every $\gamma \in G \backslash \{e\}$, there exists an epimorphism $\phi\colon G \to H$ such that H has property P and $\gamma \notin ker(\phi)$.

The following definitions should make sense once the above definition is understood.

Definition 4.2. G is said to be *residually finite* (RF) if for each $\gamma \in G \backslash \{e\}$, there is a finite-index subgroup G' such that $\gamma \notin G'$.

Now, if S is a subgroup of G, we say that G is S-RF if for each $\gamma \in G \backslash S$, there exists a finite-index subgroup G' such that $S \subset G'$ and $\gamma \notin G'$. If this condition holds, we say that G' *separates* S from γ. This is, by the way, equivalent to saying that S is the intersection of all finite-index subgroups of G containing S. Another equivalent statement is that for each $\gamma \in G \backslash S$, there is a homomorphism ϕ from G to a finite group such that $\phi(\gamma) \notin \phi(S)$. Some people refer to this property as S being "closed" in the profinite topology of G. We may also say that G is residually finite if and only if it is $\{e\}$-RF.

Definition 4.3. G is said to be *extended residually finite* (ERF) if G is S-RF for all subgroups of G. G is called *locally extended residually finite* (abbreviated "LERF") if it is S-RF for all *finitely generated* subgroups S. G is called Π_c if it is S-RF for all *cyclic* subgroups S.

Many results have been proved concerning these properties. For a comprehensive study, see [1].

It is clear from the definitions that every finite group (FIN) is ERF. ERF implies LERF, which implies Π_c, which implies RF. The strongest property here, FIN, is not very interesting in the context of the Lifting Problem. The next strongest property, ERF, seems to be a bit too strong, also. In fact, it is known that every free group of rank greater than 1 is non-ERF. In other words, for most surfaces with boundary, their fundamental groups are not ERF.

This brings us to the next property, LERF, otherwise known as subgroup separability. It seems to be an appropriate property to study since every compact surface group is finitely generated, and these are the subgroups we would like to separate in $\pi_1(M)$. The next property Π_c does not work as well if we want to consider surface groups embedded in 3-manifold groups. RF is obviously too weak for the Lifting Problem although many results are known about this property (e.g., see [14]). In particular, it is known that if the Geometrization Conjecture is true, then for every irreducible 3-manifold M, $\pi_1(M)$ is RF [27]. Hence, I suppose that if someone wants to find a counter-example to this well-known conjecture by Thurston, a good approach may be to find a non-RF group which is the fundamental group of some compact irreducible 3-manifold (although I doubt that this is possible).

One interesting fact is that if G is a finitely generated RF group, then G has solvable word problem and $Aut(G)$ too is a RF group. (Similarly, LERF implies solvable *generalized* word problem.) Another interesting question is the preservation of these properties under the direct-product and free-product operations. It turns out that the RF, Π_c, and LERF properties are preserved under the free-product operation while the FIN and ERF properties are not. Even more interesting is the fact that the direct-product operation preserves all of these properties except LERFness [22]. This is to say that the direct product of two LERF groups is not necessary LERF. See [1] and [7] for more detail. For more results on these group-theoretical concepts, see [3], [4], [6], [9], [13], [19], [20], [22], and others.

Although the LERF property provides a sufficient condition for the Lifting Problem, it is not a necessary condition. In this sense, this property may be a little too strong. In fact, it is obvious that if $\pi_1(M)$ is LERF, then *every* π_1-injective immersion lifts to an embedding in a finite cover. For many topological problems, all we need is one immersion that lifts to an embedding. Nevetheless, the LERF property seems to be essential and has played a significant role in the theory of 3-manifolds as we shall soon see in Section 5.

We conclude this section by two definitions that are perhaps more useful to topologists but also helpful to algebraists.

Definition 4.4. A compact irreducible 3-manifold M is a *graph manifold* if each component (under the canonical, or JSJ, decomposition) of $M\backslash\mathcal{T}$ is a Seifert fiber space (where $\mathcal{T}$ is the canonical family of tori). Each such component is called a *vertex manifold.* That is, each component is foliated by circles.

In this definition, we naturally include one Seifert fiber space with two torus boundary components glued together.

Definition 4.5. A properly immersed π_1-injective surface S in M is *virtually embedded* if it lifts to an embedded surface in some finite cover $\hat{M}$ of M.

Hence, if $\pi_1(M)$ is LERF, every π_1-injective immersed surface S is virtually embedded. This term should make it more convenient to discuss the Lifting Problem. However, one must be careful in using terms like "virtually embedded" since it may mean something else when used by others (e.g., [18]).

5. Historical Overview

It was many years ago when geometric topologists recognized the usefulness of combinatorial group theory and applied it to topology. Poincaré, Dehn, Haken, and Waldhausen are among the pioneers in such application. However, the first significant result in the context of subgroup separability of low-dimensional topological manifolds came out when, in 1933, Levi [10] proved that all surface groups are RF. Interestingly enough, almost 40 years later, Hempel [9] gave a 1-page proof of this same result. Then in 1949, Hall [8] showed that all free groups are LERF, thus proving that any compact surface with non-empty boundary has a LERF fundamental group (and so do handlebodies).

Then, a truly significant result came in 1978. By using topological and geometric approach, Scott [24] proved that all surface groups are LERF. Scott proved this elegant theorem as follows. First, he showed that the fundamental group of the orientable genus-2 surface is LERF. (This is by no means trivial.) Since every subgroup of a LERF group is LERF, and since the genus-2 surface is covered by every orientable surface, it follows that every (orientable) surface group is LERF. The non-orientable cases are easy as they are double covered by orientable surfaces and because finite extensions of LERF groups are LERF.

In fact, in the same paper, Scott also showed that all Fuchsian groups are LERF and so is the fundamental group of every Seifert fiber space. Considering the fact that 6 out of the 8 geometries in 3 dimensions are all Seifert fibered, this covers a lot of 3-manifolds. In particular, his result immediately gives the corollary that every S^1-bundle over a surface have a LERF fundamental group. A natural question that arises here is, "Are all 3-manifold groups LERF?" The answer to this question had to wait for another 9 years. A more specific question is the LERFness of the fundamental groups of surface bundles over S^1. Scott himself addresses this issue in his paper, saying, "I am unable to decide whether the same holds for bundles over S^1 with fibre a surface. It seems quite possible that this is false" (p. 565, [24]). He was right again.

Incidentally, there was a slight error in [24], and Scott wrote an erratum [25] in 1985, correcting his mistake. In this correction, he states that the error had been immediately noted and corrected but an open erratum seemed to be necessary since there was an increasing level of speculation that all 3-manifold groups are LERF. This almost negates his earlier speculation quoted in the last paragraph. Ironically, his first speculation in the original paper turned out to be correct and the "increasing" anticipation referred to in his correction turned out to be wrong.

Two years after Scott's erratum appeared, in 1987, Burns, Karrass, and Solitar, three algebraists at York University, came up with the first 3-manifold whose fundamental group is non-LERF [5]. As Scott had originally speculated 9 years earlier, it was a surface bundle over S^1 (with one torus boundary component). Thus, it was shown that not all 3-manifold groups are LERF.

A result such as this naturally raises dozens of new questions. For example, since their example was not a closed manifold, is the fundamental group of every *closed* 3-manifold LERF? Also, the example given in [5] is not a knot group, suggesting that perhaps all knot groups may be LERF (this was a question which was not answered until 2001 in [21]. More importantly, since LERFness is a sufficient condition and not a necessary condition, is it still possible to lift immersions to embeddings in their example manifold? This question was studied in [15], and the answer came with the help of another surprising theorem in [23] (see below). Meanwhile, more non-LERF 3-manifold groups were found [13], and in 1997 non-LERF 3-manifolds admitting a cubing of non-positive curvature were proved to exist [17]. All of these results, however, were based on the one example given in [5].

Note that all these articles addressed the non-LERFness of the fundamental groups and not the lifting of immersed surfaces. The problem of virtual embeddings, it turned out, required a more topological or geometric idea, or additional restrictions. For example, if we consider only totally geodesic immersed surfaces in hyperbolic 3-manifolds, the answer is *yes* as shown by Long [11] in 1987. Another example of additional restrictions is the cubing of non-positive curvature: under this condition, it can be shown that certain canonical surfaces in the cubing can be lifted to embeddings [2]. Some related results were also proved using a rather tedious method of "record-keeping" in the vertex manifolds of graph manifolds [16].

A more general (and perhaps a very surprising) solution came in 1998, when Rubinstein and Wang [23] gave the first argument to show why some π_1-injective immersions could not be lifted to an embedding in any finite cover of M when M is a graph manifold. In fact, their result was better: they gave a complete characterization for horizontal immersed surfaces in graph manifolds to be virtually embedded. This was the first proof that non-virtually-embedded surfaces existed in 3-manifolds (although such existence had been suspected from the non-LERFness of their fundamental groups). It turned out that the manifold produced by Burns, Karrass, and Solitar *does* contains non-virtually-embedded surfaces [15]. The argument of [23] can also produce many other examples like that. Hence, the long-awaited solution to the Lifting Problem was found, and the answer was, "Not always."

6. Recent Development

One of the questions referred to earlier was the LERFness of knot groups. Although not directly addressing this issue, Wise [28] in 1998 proved that the fundamental groups of the complements of the figure-8 knot, Whitehead link, Borromean rings, and Turk's head knot 8_{18} are all S-RF for geometrically finite subgroups S. In this paper, he refers to the well-known conjectures:

Conjectures 6.1.
(1) Every hyperbolic 3-manifold group is LERF,
(2) Every geometrically finite subgroup of a hyperbolic 3-manifold group is separable (virtually embedded), and
(3) Every Kleinian group is LERF.

These conjectures should provide a good goal for "low-dimensional topology of tomorrow."

In the same paper, Wise refers to the "Engulfing Property," originally discussed by Long [12]. (This is the property that says, "for each finitely generated subgroup $H \subset G$, there is a finite-index subgroup $G' \subset G$ containing H.") Wise showed that the Burns-Karrass-Solitar example fails to have the Engulfing Property (thus providing another proof that the group is not LERF).

One more paper worth mentioning here, dealing with similar results, is the 1999 article by Gitik [6], who defined RF and LERF in terms of profinite topology and showed that the fundamental groups of many hyperbolic 3-manifolds (with boundary) are indeed LERF.

The historical development in the previous section may give the idea that if one wants to find a strange thing, a good place to look is graph manifolds. Indeed, many non-trivial counter-examples to LERFness and virtual embeddings have been found in graph manifolds. For example, in 2001, Neumann [18] showed that there exist closed graph manifolds containing no π_1-injective immersed surfaces of negative Euler characteristics. In fact, in the same paper Neumann also showed closed graph manifolds such that no finite cover of it can contain any embedded surfaces of negative Euler characteristics. According [18], the latter collection of graph manifolds is a proper subset of the former collection.

Another significant paper appeared in 2001. This one, by Niblo and Wise [21], provides the negative answer to the question raised by Burns, Karrass, and Solitar [5] concerning the LERFness of all knot and link groups. Here, they show that if K is the sum of any non-trivial torus knot and any other non-trivial knot, then $\pi_1(S^3 \backslash K)$ is non-LERF. In particular, the square-knot complement has a non-LERF fundamental group. Depending heavily on the argument of [5], they also show that the complement of the chain of 4 circles has a non-LERF fundamental group as well. They refer to this fundamental group as L. In fact, they go even further to obtain a very interesting (and somewhat strange) result: they prove that, for graph manifolds M, $\pi_1(M)$ is LERF *if and only if* L is not a subgroup of $\pi_1(M)$. This is admittedly a very bizarre result. What it amounts to is that L is like a "common poison" found in every group that does not have this desired property called LERFness. In fact, at the present, every non-LERF 3-manifold group has this common poison L in it. Perhaps another (and probably less tasteful) way to illustrate this is that L is like the "trouble kid." A trouble kid is a child that is seen whenever and wherever there is trouble in and around school. In trying to find non-LERF 3-manifold groups, we have thus far found this trouble kid L in every known example. L is therefore the "bad guy" preventing $\pi_1(M)$ from being LERF. It is the "fly in the ointment" (Ecclesiastes 10: 1). As far as the graph-manifold groups are concerned, if the ointment $\pi_1(M)$ does not have the fly L in it, it is LERF, according to [21]. We have yet to find out if there are other such "poison" subgroups around. I do not see any particular reason that this L is such a special group. Thus, I speculate that there are other subgroups that cause 3-manifold groups to be non-LERF.

Hence, although the Lifting Problem has been answered in various manifolds and different cases, it is far from being completely solved. The same is true with the LERF property of 3-manifold groups. Along with Conjectures 2.2 and Conjectures 6.1, there are many other questions that still need to be attacked and resolved. It is my desire that the low-dimensional topology of tomorrow will build itself on the rich history, which I have tried to survey in this paper, take on these challenges, and provide some insightful and beautiful solutions.

References

[1] R. B. J. T. Allenby and R. J. Gregorac, *On Locally Extended Residually Finite Groups*, Conference on Group Theory, Lecture Notes in Mathematics **319** (1973), 9-17.

[2] I. R. Aitchison, S. Matsumoto, and J. H. Rubinstein, *Immersed surfaces in cubed manifolds*, Asian J. Math. **1** (1997), 85-95.

[3] G. Baumslag, *On the Residual Finiteness of Generalized Free Products of Nilpotent Groups*, Trans. Amer. Math. Soc. **106** (1963), 192-209.

[4] A. M. Brunner, R. G. Burns, and D. Solitar, *The Subgroup Separability of Free Products of Two Free Groups with Cyclic Amalgamation*, Contemp. Math. **33** (1984), 90-115.

[5] R. G. Burns, A. Karrass, and D. Solitar, *A Note on Groups with Separable Finitely Generated Subgroups*, Bull. Austral. Math. Soc. **36** (1987), 153-160.

[6] R. Gitik, *Doubles of Groups and Hyperbolic LERF 3-Manifolds*, Annals of Math. **150**(1999), 775-806.

[7] R. Gitik and E. Rips, *A Necessary Condition for $A = B$ to Be LERF*, Israel J. of Math. **73** (1991), 123-125.

[8] M. Hall, *Coset Representations in Free Groups*, Trans. Amer. Math. Soc. **67** (1949), 421-432.

[9] J. Hempel, *Residual Finiteness of Surface Groups*, Proc. Amer. Math. Soc. **32** (1972), 323.

[10] F. Levi, *Über die Untergruppen der Freien*, Math. Zeit. **37** (1933), 90-97.

[11] D. Long, *Immersions and Embeddings of Totally Geodesic Surfaces*, Bull. London Math. Soc. **19** (1987), 481-484.

[12] D. Long, *Engulfing and Subgroup Separability for Hyperbolic Groupss*, Trans. Amer. Math. Soc. **308** (1988), 849-859.

[13] D. Long and G. A. Niblo, *Subgroup Separability and 3-Manifold Groups*, Math. Zeit. **207** (1991), 209-215.

[14] W. Magnus, *Residually Finite Groups*, Bull. Amer. Math. Soc. **75** (1969), 305-315.

[15] S. Matsumoto, *A 3-Manifold with a Non-Subgroup-Separable Fundamental Group*, Bull. Austral. Math. Soc. **55** (1997), 261-279.

[16] S. Matsumoto, *Separability Criterion for Graph-Manifold Groups*, Topol. Appl. **93** (1999), 17-33.

[17] S. Matsumoto, *Non-Separable Surfaces in Cubed Manifolds*, Proc. Amer. Math. Soc. **125** (1997), 3439-3446.

[18] Walter Neumann, *Immersed and Virtually Embedded π_1-Injective Surfaces in Graph Manifolds*, Alg. and Geom. Topol. **1** (2001), 411-426.

[19] G. A. Niblo, *H.N.N. Extensions of a Free Group by $\mathbb{Z}$ Which Are Subgroup Separable*, Proc. London Math. Soc. **61** (1990), 18-32.

[20] G. A. Niblo, *Separability Properties of Free Groups and Surface Groups*, J. Pure Appl. Algebra **78** (1992), 77-84.

[21] G. A. Niblo and D. T. Wise, *Subgroup Separability, Knot Groups, and Graph Manifolds*, Proc. Amer. Math. Soc. **129** (2001), 685-693.

[22] E. Rips, *An Example of a Non-LERF Group Which Is a Free Product of LERF Groups with an Amalgamated Cyclic Subgroup*, Israel J. Math. **70** (1990), 104-110.

[23] J. H. Rubinstein and S. Wang, *On π_1-Injective Surfaces in Graph Manifolds*, Comm. Math. Helv. **73** (1998), 499-515.

[24] P. Scott, *Subgroups of Surface Groups Are Almost Geometric*, J. London Math. Soc. (2), **17** (1978), 555-565.

[25] P. Scott, *Correction to 'Subgroups of Surface Groups Are Almost Geometric'*, J. London Math. Soc. (2), **32** (1985), 217-220.

[26] P. Scott and C. T. C. Wall, *Topological Methods in Graph Theory*, Homological Group Theory: Proceedings of a Symposium held at Durham in September 1977, ed. by C. T. C. Wall, Cambridge Univ. Press, London Math. Soc. Lecture Series **36** (1979), 137-203.

[27] W. Thurston, *Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry*, Bull. Amer. Math. Soc. **6** (1982), 357-381.

[28] D. T. Wise, *Subgroup Separability of the Figure-8 Knot Group*, preprint, 1998.

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